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LETTER TO THE EDITOR

A non-equilibrium percolation transition in random Ising ferromagnets

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Abstract. We study the geometrical properties of the domain structure that results when a random Ising ferromagnet is quenched to zero temperature. We define a novel connectivity transition between like spins in the frozen configuration, and show that it is in a new universality class, distinct from that of random percolation. The scaling is robust and independent of model details. The relevance of this transition to spinodal decomposition is discussed.

This letter builds on three well known themes but in a new non-equilibrium context. Percolation [1] and phase separation [2] are two widely studied and well understood problems in statistical mechanics. The former deals with the issue of geometrical connectivity [3] and finds applications in a variety of situations including transport in random composites [4], nonlinear flow on rough surfaces [5], fracture [6] and dilute magnets [7]. At the connectivity threshold, one obtains scale-invariant behaviour characterized by critical exponents with scaling relations between them. Furthermore, the exponents are universal and do not depend on microscopic details such as the inclusion of next-nearest-neighbour connectivity or the nature of the lattice. The second, phase separation, is most simply illustrated by considering an Ising ferromagnet (which is in the same equilibrium universality class as a binary alloy or a binary mixture of fluids) with a random initial condition quenched to a temperature below the critical point. The spin configuration evolves with time in such a way as to reach a state of minimum free energy—in a disordered magnet at zero temperature, the system attempts to reach its ground state but may not succeed in doing so because of the existence of local minima in the energy landscape in which it may get caught [8]. The third theme is a general principle in equilibrium statistical mechanics enunciated by Berker and others [9] which states that symmetry-breaking first-order transitions in two dimensions are converted to second-order transitions even by an infinitesimal bond randomness [10].

In this letter we combine ideas from percolation and phase separation to define a dynamical version of the site percolation problem. We investigate the geometrical properties of the frozen domain structure that results when a disordered ferromagnet is quenched to zero temperature. One can define a connectivity transition between clusters of like spins in the frozen state, as the fraction of up spins, in the initial condition, is varied. Surprisingly,

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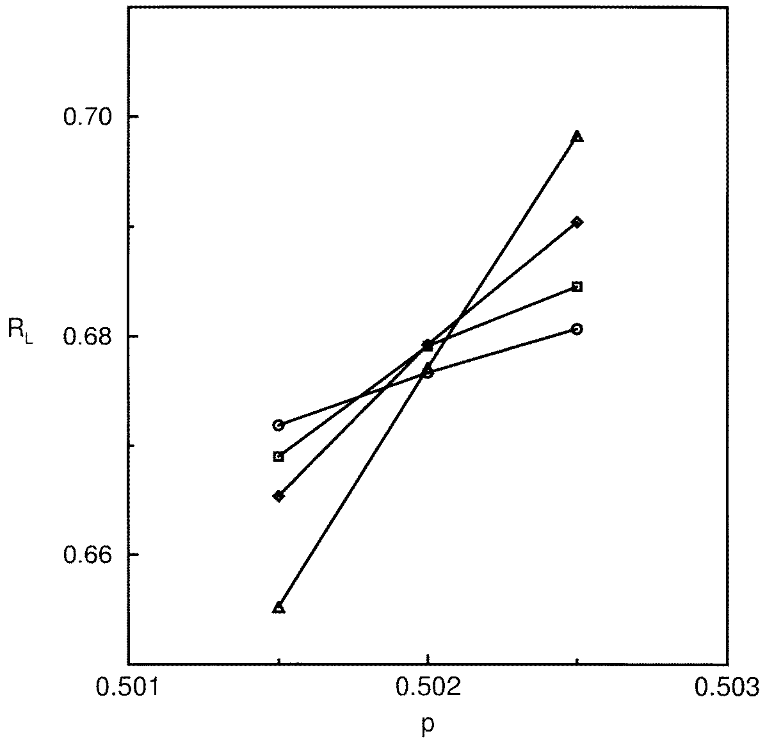

Figure 1. The fraction of percolating samples, $R_L(p)$, on a square lattice as a function of p and for different values of L. The intersection of the curves gives the percolation threshold $p_c = 0.5020 \pm 0.0003$. The system sizes are: (○) $L = 32$, (□) $L = 64$, (◊) $L = 128$ and (△) $L = 256$.

in 2D, the critical threshold is found to be exceedingly close to $\frac{1}{2}$, i.e. a symmetric quench, and the transition is shown to be in a different universality class than random percolation. The new critical exponents are robust and insensitive to model details. Our findings raise questions about some of the underlying assumptions of scaling theories of phase separation in disordered systems.

Consider an Ising ferromagnet with a random initial condition in which a fraction p of the spins, say, is up and the rest point down. Ordinary random percolation would consist of studying the connectivity threshold of the up spins on changing p. We now perform a quench to zero temperature using single-spin-flip Glauber dynamics. A spin is chosen at random and flipped only if the energy of the system is not increased. This procedure is repeated until every spin is aligned with its local field; the system is then in a local energy minimum. As we vary p (measured in the initial state before quenching), each time starting from a new random initial condition, we seek a threshold value corresponding to the onset of connectivity of the up spins in the configuration obtained after the quench dynamics has ceased.

For a ferromagnet with uniform nearest-neighbour exchange, the above procedure gives rise to a 'first-order' transition at $p = \frac{1}{2}$. The system is able to reach one or the other of its two fully aligned ground states depending on whether $p < \frac{1}{2}$ or $p > \frac{1}{2}$. Thus, the percolation probability is strictly either 0 or 1. However, if bond randomness is introduced into the model, the ground state is not reached and the system remains in a metastable

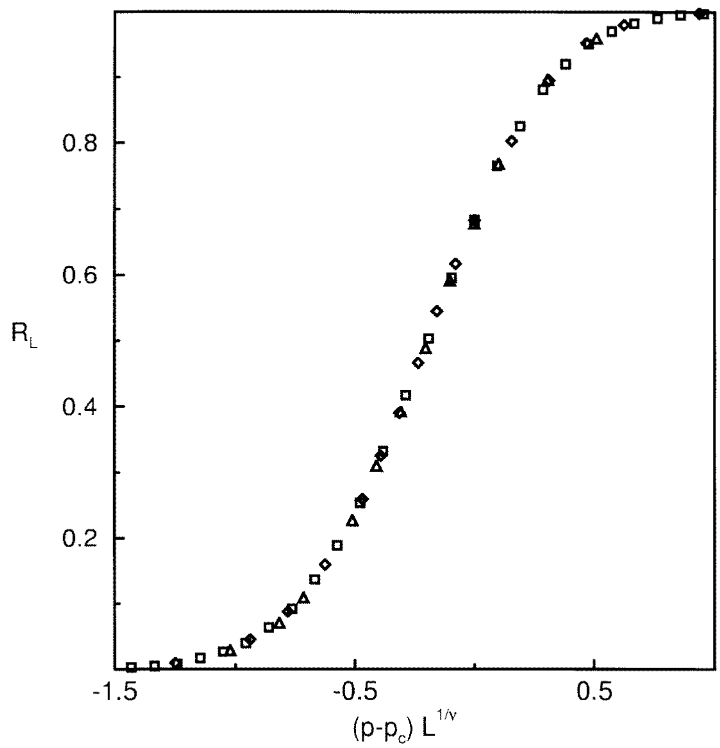

Figure 2. Finite-size collapse for $R_L(p)$ on a triangular lattice with $p_c = \frac{1}{2}$ and the exponent $\nu = 1.41$. System sizes are: (□) $L = 64$, (◊) $L = 128$ and (△) $L = 256$.

configuration comprising interpenetrating clusters of up and down spins. In analogy with the Berker prediction [9], the connectivity transition upon varying p becomes continuous. Of course, this is only a loose analogy because here the final states are not sampled according to an equilibrium probability distribution and thus the arguments of [9] do not hold in a straightforward manner.

Our analysis of this transition is based on a finite-size scaling [11] study of the frozen spin configurations in a random bond Ising ferromagnet, with couplings chosen from a uniform distribution between 0 and 1. We shall focus on determining the connectivity threshold, p_c, and the exponents ν, β and γ which, in an infinite system, characterize the singular behaviour of the correlation length, ξ, the order parameter, P_∞, and the susceptibility, χ_∞ [1]:

$$\xi \sim (p - p_c)^{-\nu} \qquad P_\infty \sim (p - p_c)^{\beta} \qquad \chi_\infty \sim (p - p_c)^{-\gamma}. \tag{1}$$

We define $R_L(p)$ to be the probability of finding a spanning cluster in a system of size L. As characteristic of critical phenomena and standard percolation, $R_L(p)$ has a scaling form

$$R_L(p) = f[(p - p_c)L^{\frac{1}{\nu}}] \tag{2}$$

which is the consequence of there being a single dominant length scale which diverges at p_c. Plots of $R_L(p)$ against p for a square lattice with L ranging from 32 to 256 are shown in figure 1. These curves were calculated numerically by averaging over up to 10^6 different realizations of the frozen configurations, for a range of values of p. The crossing point accurately determines the percolation threshold $p_c = 0.5020 \pm 0.0003$ which is very close,

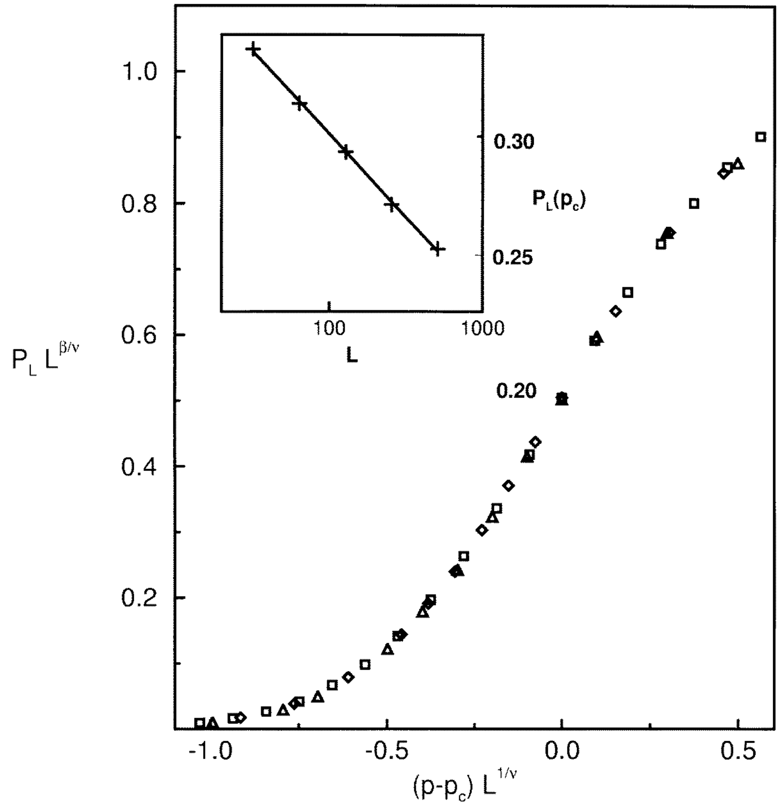

Figure 3. Collapse of the order parameter $P_L(p)$ for a triangular lattice with $p_c = \frac{1}{2}$. The exponents are $\nu = 1.41$ and $\beta = 0.155$. The linear dimensions L are: (□) $L = 64$, (◇) $L = 128$, (△) $L = 256$. The inset shows a log–log plot of $P_L(p)$ at $p = p_c$ as a function of L, giving $\frac{\beta}{\nu} = 0.109$.

but demonstrably different, from $\frac{1}{2}$. We have repeated the above analysis for spins on a square lattice with both nearest- and next-nearest-neighbour couplings and connectivity. In this case we find $p_c = 0.498 \pm 0.001$, again close to $\frac{1}{2}$.

We note that for the triangular lattice $p_c = \frac{1}{2}$ exactly. Indeed the percolation threshold in the quenched state, p_c, satisfies $1 - p_c^r \leqslant p_c \leqslant p_c^r$, where $p_c^r \geqslant \frac{1}{2}$ is the threshold for random percolation (regular site percolation). This follows from the observation that when $p > p_c^r$, the dynamics do not destroy the connectivity implying that $p_c \leqslant p_c^r$. Likewise, when $p < 1 - p_c^r$, the dynamics do not break the connectivity of the down spins so that $p_c \geqslant 1 - p_c^r$. For the triangular lattice $p_c^r = \frac{1}{2}$ and therefore p_c is also $\frac{1}{2}$.

Once p_c has been determined, the critical exponent ν can be estimated from a scaling collapse of the full set of data. We will present results for the triangular lattice where p_c is known exactly. The best collapse of $R_L(p)$ is shown in figure 2, giving an exponent $\nu = 1.41 \pm 0.02$. Note that this is different from the value for random percolation, $\nu = \frac{4}{3}$ [12]. We have verified that our method of analysis reproduces this exact value in the random case without dynamics where we find $\nu = 1.33 \pm 0.03$.

The order-parameter exponent β can be determined in a similar way by considering the scaling behaviour of $P_L(p)$, the probability that a randomly chosen spin belongs to the

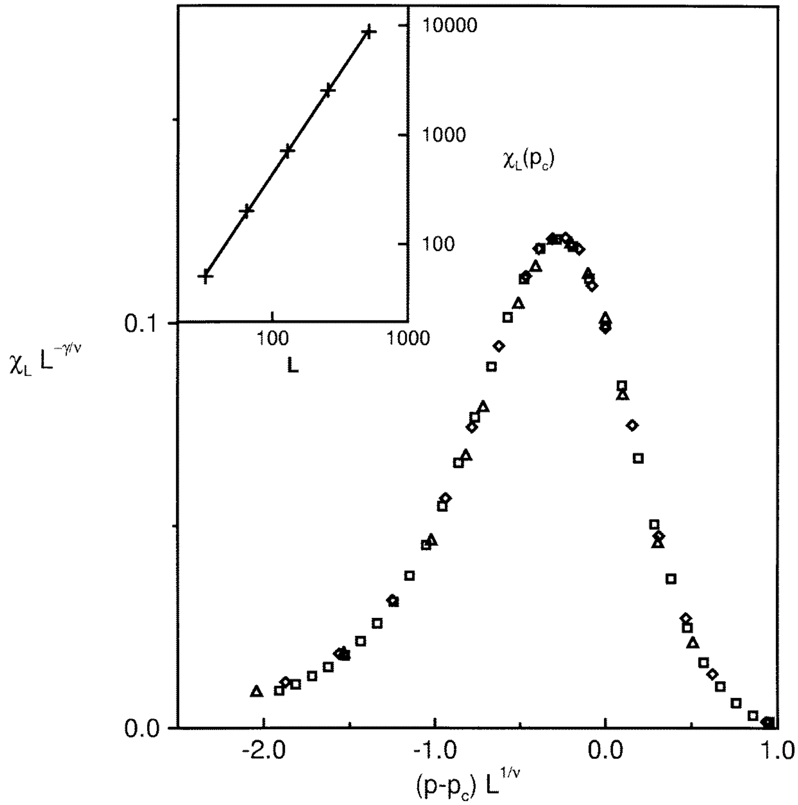

Figure 4. Collapse of the susceptibility $\chi_L(p)$ for a triangular lattice with $p_c = \frac{1}{2}$. The exponents are $\nu = 1.41$ and $\gamma = 2.58$. The linear dimensions L are: (□) $L = 64$, (◇) $L = 128$, (△) $L = 256$. The inset shows a log–log plot of $\chi_L(p)$ at $p = p_c$ as a function of L, giving $\frac{\gamma}{\nu} = 1.85$.

infinite cluster. It can be written in the scaling form

$$P_L(p) = L^{-\frac{\beta}{\nu}} g[(p - p_c)L^{\frac{1}{\nu}}] \tag{3}$$

and results for the optimal scaling collapse are shown in figure 3. We find $\beta = 0.155 \pm 0.01$ with $\nu = 1.41$ as determined above. The inset is a log–log plot of $P_L(p)$ at $p = p_c$ as a function of L, which should decay as $L^{-\frac{\beta}{\nu}}$ from (3). We observe a linear dependence for systems of size up to $L = 512$ which gives $\frac{\beta}{\nu} = 0.109 \pm 0.002$. This is consistent with the exponent values determined from the data collapse and, furthermore, the data shows no deviation from simple linear behaviour. The measured value of β is also different from that of random percolation ($\beta = \frac{5}{36}$).

In addition, we have determined the susceptibility exponent γ, which, in standard percolation theory, is related to ν and β through the hyperscaling relation [1]

$$d\nu = 2\beta + \gamma \tag{4}$$

where d is the spatial dimension. The susceptibility $\chi_L(p)$ is defined by the second moment of the cluster size distribution, excluding the spanning cluster. It has a scaling form

$$\chi_L(p) = L^{\frac{\gamma}{\nu}} h[(p - p_c)L^{\frac{1}{\nu}}] \tag{5}$$

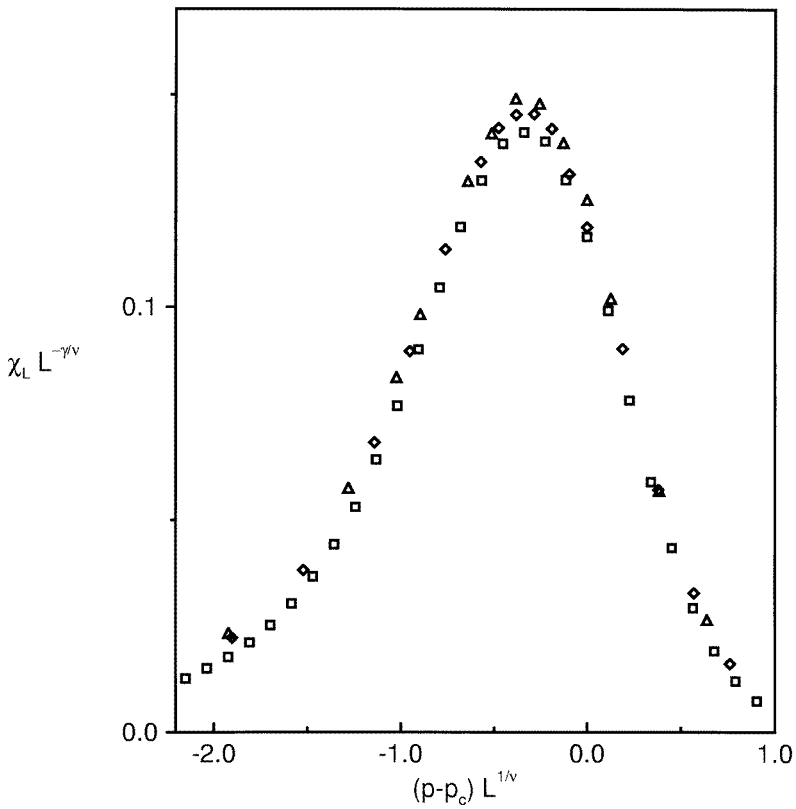

Figure 5. An attempt at a scaling collapse of the susceptibility $\chi_L(p)$ for a triangular lattice with $p_c = \frac{1}{2}$ and the exponents taking their standard percolation values, $\nu = \frac{4}{3}$ and $\gamma = \frac{43}{18}$. The linear dimensions L are: (□) $L = 64$, (◇) $L = 128$, (△) $L = 256$.

and diverges at p_c in the thermodynamic limit. Figure 4 shows a collapse of $\chi_L(p)$ with the best estimate of $\gamma = 2.58 \pm 0.03$, with $\nu = 1.41$. The inset is a log–log plot of $\chi_L(p_c)$ against L, from which we determine $\frac{\gamma}{\nu} = 1.85 \pm 0.02$. This is again consistent with the scaling collapse. The value of γ is also different from that of standard percolation, $\gamma = \frac{43}{18}$. However, it is consistent with the hyperscaling relation, within statistical errors, when ν and β take their new values. These results suggest that the dynamical percolation transition studied here is in a new universality class. Indeed, we have tried the above scaling analysis with the exponents taking their values for standard percolation. Figure 5 shows an attempt to collapse the susceptibility data using $\nu = \frac{4}{3}$, $\gamma = \frac{43}{18}$. The quality of the collapse is much lower than that with the new exponents, as shown in figure 4.

As a test of the robustness of this scaling behaviour, we have considered the effects of changing the microscopic details of the system. In particular, we have repeated the above scaling analysis for Ising spins on a square lattice with nearest-neighbour interactions, and on a square lattice with both nearest- and next-nearest-neighbour interactions and connectivity. In all cases we find values of the critical exponents which are the same as those determined from the triangular lattice, and different from those of standard percolation.

Our findings in the context of a percolation transition are also of importance to the problem of phase ordering in disordered systems. It is well known that in a random Ising ferromagnet at a non-zero temperature, the characteristic domain size grows logarithmically with time due to thermally activated motion of the interfaces. However, there is an apparent discrepancy between the predictions of scaling theory [14], which assumes local equilibrium, and that of numerical simulations [8]. At very early times after a quench the spins will reach the states sampled in our simulations. Thus, for a symmetric quench, the system will be extremely close to the percolation critical point. One would then expect that, in the presence of thermal fluctuations, non-trivial dynamical scaling could result. A further consequence of a quench to the percolation critical point is that the spanning clusters will have fractal scaling properties. This provides a qualitative explanation for the non-trivial scaling of the interfaces discussed in [13]. Moreover, the fractal dimension determined in [13] is also different from the corresponding hull dimension of random percolation, providing additional evidence for a new percolation universality class.

To summarize, we have defined a dynamical connectivity transition which is in a new percolation universality class. The critical behaviour is robust and independent of the microscopic details of the model, and the standard hyperscaling relation is seen to hold. Our numerical results are, of course, subject to size limitations and it is not possible to entirely rule out a return to the regular percolation universality class for sizes beyond the scope of our investigation. However, the existence of a dynamically accessible critical point and its connection to the fractal interfaces observed in [13] suggest that this is not the case. It is also interesting that the replacement of first-order transitions by continuous ones due to bond randomness in equilibrium symmetry-breaking transitions have an analogue in the dynamical non-equilibrium transition that we have studied here.

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